Quadratic Convergence

Let x_n be a sequence that converges to s. Let $e_n = x_n - s$. We say the sequence *converges quadratically* if there is a constant c so that $|e_{n+1}| \leq c|e_n|^2$. Then the following estimate is true:

$$|e_n| \leq \frac{1}{c}|ce_0|^{2^n}.$$

Proof. The assumption can be written

$$|e_{n+1}| \leq \frac{1}{c}|ce_n|^2.$$

We prove the statement by induction on n. It is true for $n = 0$, so assume it is true for n. Then

$$|e_{n+1}| \leq \frac{1}{c}|ce_n|^2 \leq \frac{1}{c}[|ce_0|^{2^n}]^2 = \frac{1}{c}|ce_0|^{2^{n+1}}.$$

□